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Analytical expression for zero Lyapunov exponent of chaotic noised oscillators

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ABSTRACT

This paper is devoted to the analytical formula for zero Lyapunov exponent describing the dynamics of interacting chaotic systems with noise. The deduced analytical prediction is in a good agreement with the value of zero Lyapunov exponent obtained numerically for two unidirectionally coupled Rössler oscillators. We have shown that this good agreement is observed for a wide diapason of the values of the control parameters.

1. Introduction

There are no doubts that Lyapunov exponents (LEs) are very powerful tool used frequently to study the complex dynamics of nonlinear systems. The theory and procedures of Lyapunov exponent calculation have been properly developed [1–4] and now Lyapunov exponents are used widely in very different fields of science, including (but not limited to) physics [5], astronomy [6], medicine [7], economy [8], etc. Due to their great efficiency Lyapunov exponents are applied to a large number of complex systems, including spatially extended ones [9–15]).

Nowadays, among techniques devoted to the Lyapunov exponent calculation for nonlinear systems two main approaches may be distinguished generally, namely, (i) the estimation of the largest Lyapunov exponents from time series [4] of the examined system and (ii) the application of the standard procedures [3,4] based on the numerical calculation of Lyapunov sums with the help of the system evolution operator and its linearization. Estimation of the largest Lyapunov exponent from time series is very important for experiments (when the evolution operator is unknown) and used widely for the experimental data, including data of living systems (see, e.g., [16]). The second approach allows to calculate a spectrum of Lyapunov exponents but it requires an explicit form of the evolution operator of the system under study.

The very interesting, important and promising point is the analytical estimation of Lyapunov exponents. The analytical formula for the value of Lyapunov exponent can be rather easily obtained only for the steady-state solutions, namely, for the fixed points of nonlinear systems with a small number of degrees of freedom and for the steady-state spatially homogeneous solutions of the spatially extended systems. More interestingly, the value of Lyapunov exponent has been

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obtained analytically for simple models of nonlinear oscillators being under noise [17–20] (typically, in connection with the problem of synchronization of ensemble of oscillators by common external noise) as well as for the neural oscillator models [21–23].

The next step in the use of the Lyapunov exponent apparatus is the problem concerning the analytical description of LEs for chaotic oscillators. At present time in certain cases for such systems the analytical approximations of Lyapunov exponent were obtained, but these approximations remain valid in a very narrow diapason of the values of the control parameters [24,25] and do not provide a complete picture of the LE behavior.

In this paper we report for the first time on the analytical formula for the so-called zero Lyapunov exponent (ZLE) of chaotic dynamical systems with noise. ZLE stands out among the spectrum of LEs which characterize the complex behavior of chaotic systems. Zero Lyapunov exponent exists necessarily in the LE spectrum of the flow systems characterizing the evolution of the perturbation along the phase trajectory. For two coupled flow oscillators (which possesses two zero LEs in the case without coupling) one of ZLEs diverges from the zero value with the growth of the coupling strength. In addition, ZLE corresponds to the leading Lyapunov exponent of the phase oscillator models [17–20]. It is LE to be the object of the main interest in the present work. ZLE plays a crucial role in some relevant circumstances, e.g., in the synchronization phenomena. In particular, transition of one of ZLEs into the negative value region is related to the phase synchronization phenomenon [26,27], although the transition point does not coincide with the phase synchronization boundary [24,28]. ZLE may also be an indicator of the peculiar regimes of the system behavior, e.g., the incomplete noise-induced synchronization [29].

In parallel with the coupling strength, noise also influences on the LEs. Noise is observed in experimental studies as well as in numerical simulations. Typically, the influence of noise is crucial for the system dynamics (see e.g., Ref. [30–33]).

In addition, the certain phenomena take place in systems with both deterministic (but chaotic) and stochastic dynamics. Indeed, for the driven periodic oscillator behavior the synchronization is known to be connected with the saddle-node bifurcation [34]. The same scenario takes place for chaotic oscillators being in the phase synchronization regime, although its manifestation is hidden due to the aperiodic motion [28,35]. As a consequence, the phenomena observed near the synchronization boundary of periodic oscillators whose dynamics is perturbed by noise have been shown recently to be the same as for the chaotic systems being close to the phase synchronization onset [25,28,32,36]. Similarly, the noise-induced synchronization and generalized synchronization are caused by one and the same mechanism, with the difference between them being only in the driving signal [37]. All findings mentioned above mean that in certain cases the dynamics of chaotic systems may be modeled by the behavior of the periodic systems perturbed by noise. Therefore, the results given in this paper concerning the analytical expression for Lyapunov exponent may be applicable both for the stochastic and deterministic systems as well as for the deterministic systems with noise.

2. Theoretical background

To obtain analytical expression for zero Lyapunov exponent of coupled chaotic oscillators one have to take into account the following points: (i) under certain conditions chaotic oscillators may be modeled by a noised periodic oscillator (see, e.g., [25,37,38]) and (ii) for periodically driven nonlinear oscillator the boundary of synchronization is described by the saddle-node bifurcation [34], with the very same mechanism (but, masked by the irregular dynamics) taking place both for the periodic oscillator perturbed by the external noise and for the chaotic system [25,28]. In other words, to get analytical expression for ZLE, one can consider a model system describing the behavior of driven periodic oscillator with noise in the vicinity of the synchronization onset. From this point of view, the circle map [39–41]

$$\varphi_{n+1} = \varphi_n + \Omega + \varepsilon f(\varphi_n) + \xi_n, \quad \mod 2\pi \tag{1}$$

being a classical model to study nonlinear phenomena [42–44] including synchronization [25] and phase locking [45,46] is the very suitable dynamical system to estimate the value of ZLE of driven periodical oscillator with noise as well as the chaotic oscillators. The circle map (1) is known to describe very precisely the behavior of driven periodical isochronous oscillator near the synchronization onset, since it is, in fact, the discretization of Adler's equation [47] or truncated equation [34,48] deduced in the framework of the complex amplitude method. Tangential bifurcation taking place in the circle map (1) without noise corresponds to the saddle-node bifurcation in the truncated equation and to the synchronization onset of the driven periodical oscillator [34], respectively. The added noise term in (1) enhances the application of the considered map to the noised periodical oscillators and chaotic systems. In Eq. (1) $\varphi \in [0, 2\pi)$ is an angle, parameter $\varepsilon \geq 0$ is a measure of the strength of nonlinearity, ξ_n is a delta-correlated Gaussian noise $[\langle \xi_n \rangle = 0, \langle \xi_n \xi_m \rangle = D\delta(n-m)]$, $f(\varphi) = \sin\varphi$. In fact, sin-circle map (1) describes the noised dynamical oscillators driven by an external force whose frequency and amplitude are described by the dimensionless parameters Ω and ε, respectively.

The Lyapunov exponent Λ_0 of circle map (it corresponds to the zero Lyapunov exponent of the coupled flow systems) is

$$\Lambda_0 = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln|1 + \varepsilon f'(\varphi_i)|, \tag{2}$$

where $\{\varphi_n\}$ is the time sequence of system (1). Having based on the ergodicity of the examined process and taken into account 2π-periodicity, we can obtain

$$\Lambda_0 = \int_0^{2\pi} \rho_i(\varphi) \ln|1 + \varepsilon f'(\varphi)|\, d\varphi, \tag{3}$$

where

$$\rho_i(\varphi) = \rho_i(\varphi + 2\pi) \tag{4}$$

is the stationary probability density for φ-variable.

Having supposed that φ is changed per one iteration insignificantly, we consider $(\varphi_{n+1} - \varphi_n)$ as the time derivative $\dot{\varphi}$. Due to the stochastic term in (1) we must consider the stochastic differential equation (SDE)

$$d\Phi = (\Omega + \varepsilon \sin\Phi)\, dt + dW, \tag{5}$$

where $W(t)$ is a one-dimensional Wiener process and $\Phi(t)$ is stochastic process. In other words, the stochastic map (1) is reduced to the equation of the phase oscillator (see also works [17–20]).

To solve SDE (5) we consider Fokker–Plank equation

$$\frac{\partial\rho(\varphi,t)}{\partial t} = -\frac{\partial}{\partial\varphi}\left[(\Omega+\varepsilon\sin\varphi)\rho(\varphi,t)\right] + \frac{D}{2}\frac{\partial^2\rho(\varphi,t)}{\partial\varphi^2}, \tag{6}$$

where the probability density $\rho(\varphi,\ t)$ characterizes the stochastic process $\Phi(t)$. For the invariant probability density $\rho_i(\varphi)$ Eq. (6) takes the form

$$\frac{D}{2}\frac{d^2\rho_i(\varphi)}{d\varphi^2} - \frac{d}{d\varphi}\left[(\Omega+\varepsilon\sin\varphi)\rho_i(\varphi)\right] = 0 \tag{7}$$

and may be reduced to

$$\rho_i'(\varphi) - \frac{2}{D}\left[(\Omega+\varepsilon\sin\varphi)\rho_i(\varphi)\right] = \hat{C}, \tag{8}$$

where $\hat{C} = \text{const}$. The general solution of (8) is given by

$$\rho_i(\varphi) = A\left[\Pi\left(\varphi, \frac{2\varepsilon}{D}, \frac{2\Omega}{D}\right) + C\right] \times \exp\left(-\frac{2}{D}(\varepsilon\cos\varphi - \Omega\varphi)\right), \tag{9}$$

where

$$\Pi(\varphi,\hat{\varepsilon},\hat{\Omega}) = \int_0^{\varphi} \exp\left(\hat{\varepsilon}\cos\eta - \hat{\Omega}\eta\right)d\eta, \tag{10}$$

constant

$$C = \frac{\Pi\left(2\pi, \frac{2\varepsilon}{D}, \frac{2\Omega}{D}\right)}{\exp\left(\frac{4\pi\Omega}{D}\right) - 1} \tag{11}$$

can be found from the periodicity condition (4), whereas A is a normalizing factor providing the normalization condition for 2π-interval

$$\int_0^{2\pi} \rho_i(\varphi)\,d\varphi = 1. \tag{12}$$

Finally, having combined Eqs. (3) and (9)–(11) one can get the analytical formula for LE of the circle map which depends on the parameters ε, Ω and D as

$$\Lambda_0(\varepsilon,\Omega,D) = A\int_0^{2\pi}\left(\Pi\left(\varphi,\frac{2\varepsilon}{D},\frac{2\Omega}{D}\right) + \frac{\Pi\left(2\pi,\frac{2\varepsilon}{D},\frac{2\Omega}{D}\right)}{\exp\left(\frac{4\pi\Omega}{D}\right)-1}\right) \times \exp\left(-\frac{2}{D}(\varepsilon\cos\varphi - \Omega\varphi)\right)\ln|1+\varepsilon\cos\varphi|\,d\varphi. \tag{13}$$

To illustrate the correctness of the obtained analytical expressions we compare the theoretical predictions (Eqs. (9) and (13)) with the results of numerical calculation of the dynamics of the circle map (1). The control parameter of the system under study has been selected as $\Omega = 0.1$, $D = 0.04$.

Without noise (i.e., if $D = 0$) the saddle-node bifurcation (corresponding to the boundary of synchronization of the flow systems) is observed in (1) for $\epsilon = \epsilon_c = \Omega$, where the stable ($\varphi_s = \pi + \arcsin(\Omega/\varepsilon)$) and unstable $\varphi_u = (2\pi - \arcsin(\Omega/\varepsilon))$ fixed points annihilate at $\varphi_c = 3\pi/2$. Below the bifurcation point ($\varepsilon < \Omega$) there are no fixed points, whereas for $\varepsilon > \Omega$ two fixed points are observed in the range $[0, 2\pi)$, with the stable one corresponding to the synchronous regime in the flow systems and the unstable one being the boundary of the basin of attraction of the stable fixed point. The invariant probability density $\rho_i(\varphi)$ in the noiseless case is

$$\rho_i(\varphi) = \begin{cases} \dfrac{\sqrt{\Omega^2-\varepsilon^2}}{2\pi(\Omega+\varepsilon\sin\varphi)}, & \varepsilon < \Omega, \\ \delta(\varphi-\varphi_s), & \varepsilon > \Omega, \end{cases} \tag{14}$$

where $\delta(\cdot)$ is a Dirac δ-function. If the external noise $D > 0$ influences on the system dynamics, the invariant probability distribution is given by Eq. (9), with the maximum and minimum values of $\rho_i(\varphi)$ corresponding to the fixed points φ_s and φ_u, respectively.

In Fig. 1 the curves corresponding to the stationary probability densities $\rho_i(\varphi)$ for the different values of parameter ε are given. The solid curves represent the analytical estimations, whereas the points correspond to the data obtained by the numerical simulations of (1). The good agreement between the theoretical expressions and points obtained numerically is observed for all values of parameter ε.

Fig. 2 shows the dependence of LE Λ_0 on the coupling parameter ε as well as its theoretical prediction (13) in the

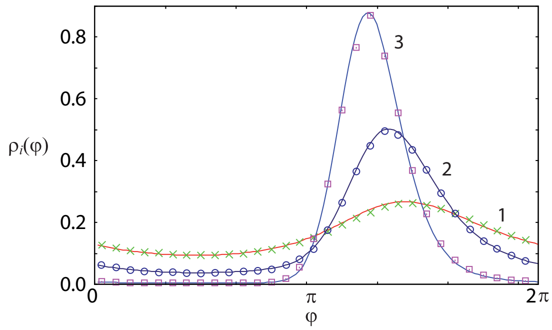

Fig. 1. The invariant probability density $\rho_i(\varphi)$ of φ-variable. The solid lines correspond to the theoretical predictions of the probability density $\rho_i(\varphi)$, whereas the numerically obtained data for (1) are given by points. The calculations have been done for $\Omega = 0.1$, $D = 0.04$, curve 1 — $\varepsilon = 0.05$, curve 2 — $\varepsilon = \varepsilon_c = 0.10$, curve 3 — $\varepsilon = 0.15$.

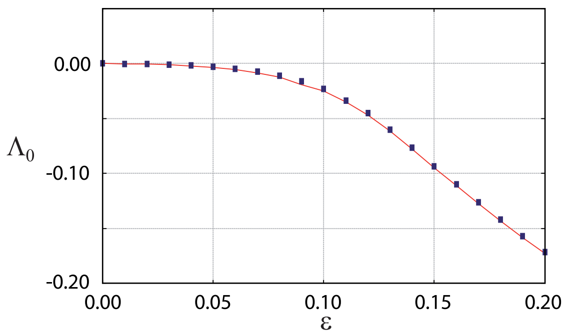

Fig. 2. The dependence of Lyapunov exponent Λ_0 on the control parameter ε for the circle map (1). The points correspond to the numerically calculated data, the solid line represents the analytical expressions for LE (13).

whole diapason of the coupling strength values. Again, one can see the very good coincidence of the theoretical curves and the numerically obtained values of LE being the convincing evidence of the correctness of the obtained analytical expression.

So, analytical predictions (9) and (13) agree well with the numerical results obtained for the circle map.

3. Two coupled Rössler systems & results of numerical calculations

Having obtained the theoretical expression for ZLE and verified it numerically with the help of the discrete map, we examine now the flow chaotic systems. Typically, the theoretical expressions deduced for the model discrete systems may be applied to the flow systems due to the flows and maps are tightly interrelated with each other [49,50]. In particular, the flow systems may be reduced to the maps with the help of Poincaré recurrence technique [51]. Since the circle map (1) considered in the previous Section is the appropriate discrete model system to describe the behavior of the driven periodic oscillator with noise and coupled chaotic systems (see also [25,28,32]), one can use Eqs. (9) and (13) to estimate the zero Lyapunov exponent value of two coupled chaotic systems.

The phase φ in Eq. (9) makes sense of the instantaneous phase difference $\Delta\varphi(t)$ between the considered flow oscillators, whereas the invariant probability density $\rho_i(\varphi)$ corresponds to the phase difference distribution $\rho(\Delta\varphi)$. To make possible the introduction of the instantaneous phases of chaotic oscillators, the chaotic systems under study should be characterized by the phase-coherent chaotic attractors [52,53]. Typically, the coupled oscillators are considered with the coupling strength σ between them being varied, with the parameter σ corresponding to ε in (1), $\sigma \sim \varepsilon$. In this case the dependence (13) may be rewritten as

$$\Lambda_0 = Af\int_0^{2\pi}\left(\Pi(\Delta\varphi, a_1\sigma, a_2) + \frac{\Pi(2\pi, a_1\sigma, a_2)}{\exp(2\pi a_2) - 1}\right) \times \exp(a_2\Delta\varphi - a_1\sigma\cos\Delta\varphi)\ln|1 + a_3\sigma\cos\Delta\varphi|\, d(\Delta\varphi). \quad (15)$$

where a_1, a_2 and a_3 are the constants whose values depend on the examined system, f is the main frequency of the drive system [53]. As far as the distribution of the phase difference between interacting oscillators $\rho(\Delta\varphi)$ is concerned (see Eq. (9)), it may be described by

$$\rho(\Delta\varphi) = A\left(\Pi(\Delta\varphi, a_1\sigma, a_2) + \frac{\Pi(2\pi, a_1\sigma, a_2)}{\exp(2\pi a_2) - 1}\right) \times \exp(a_2\Delta\varphi - a_1\sigma\cos\Delta\varphi). \quad (16)$$

As a model chaotic systems we have used two unidirectionally coupled Rössler systems

$$\begin{aligned}
\dot{x}_d &= -\omega_d y_d - z_d,\\
\dot{y}_d &= \omega_d x_d + a y_d,\\
\dot{z}_d &= p + z_d(x_d - c),\\
\dot{x}_r &= -\omega_r y_r - z_r + \sigma(x_d - x_r) + \xi,\\
\dot{y}_r &= \omega_r x_r + a y_r,\\
\dot{z}_r &= p + z_r(x_r - c),
\end{aligned} \quad (17)$$

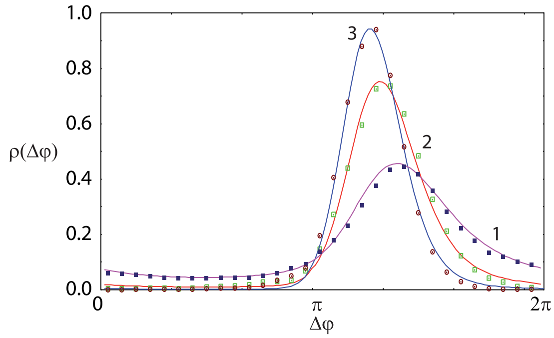

Fig. 3. The phase difference distributions $\rho(\Delta\varphi)$ of two unidirectionally coupled Rössler oscillators (17) and its theoretical predictions. The analytical expressions (16) are shown with the help of the solid lines, whereas the data obtained numerically are given by points. The coupling parameter value is $\sigma = 0.03$, (curve 1), $\sigma = 0.04$ (curve 2) and $\sigma = 0.05$ (curve 3).

where (x_d, y_d, z_d) $[(x_r, y_r, z_r)]$ are coordinates describing the drive and response oscillators, respectively, σ is parameter of the coupling strength and $\xi(t)$ is a delta-correlated Gaussian white noise $[\langle\xi(t)\rangle = 0, \langle\xi(t)\xi(\tau)\rangle = B\delta(t-\tau)]$. The control parameters $a = 0.15$, $p = 0.2$, $c = 10.0$ have been selected according to [54,55]. The ω_r–parameter determining the response system main frequency has been set as $\omega_r = 0.95$, whereas the analogous parameter of the drive system was $\omega_d = 0.93$. To integrate Eq. (17) we have used the one-step Euler method with the time step $h = 10^{-3}$.

Typically, the instantaneous phase of chaotic oscillators may be introduced with the help of different techniques [34,38,56]. In the present work the instantaneous phases $\varphi(t)_{d,r}$ of considered signals have been introduced as the rotation angles $\varphi(t)_{d,r}$ on the planes $(x_{d,r}, y_{d,r})$, i.e., $\tan\varphi_{d,r} = y_{d,r}/x_{d,r}$. Since the theoretical prediction (16) has been obtained for 2π-periodical boundary condition (4), the range of the instantaneous phase values was limited to the interval $[0; 2\pi)$, with the phase difference values of the response and drive oscillators being also reduced to 2π-interval. Finally, to compare the theoretical formula with the results of the direct numerical simulations of the noised chaotic oscillators dynamics, the obtained distributions of the phase differences $\rho(\Delta\varphi)$ have been normalized for they to satisfy the normalization condition

$$\int_0^{2\pi}\rho(\Delta\varphi)\, d(\Delta\varphi) = 1 \quad (18)$$

as it has been done for the invariant probability density $\rho_i(\varphi)$ (see Eq. (12)). Since in Section 2 the initial phase φ_0 in the nonlinear periodical function $f(\varphi) = \sin(\varphi + \varphi_0)$ has been chosen as $\varphi_0 = 0$ rather arbitrary, the phase difference distributions $\rho(\Delta\varphi)$, in general, can coincide with the theoretical prediction (16) up to a certain shift of the phase.

The comparison of the phase difference distributions $\rho(\Delta\varphi)$ obtained up to a certain phase shift discussed above with the theoretical predictions (16) is shown in Fig. 3 for the different coupling strength values, σ, and the noise intensity $B = 4 \times 10^2$. The phase difference distributions of coupled Rössler oscillators calculated numerically are given by the points, whereas the theoretical curves corresponding to Eq. (16) with parameters $a_1 = 166.7$ and $a_2 = 5.0$ are

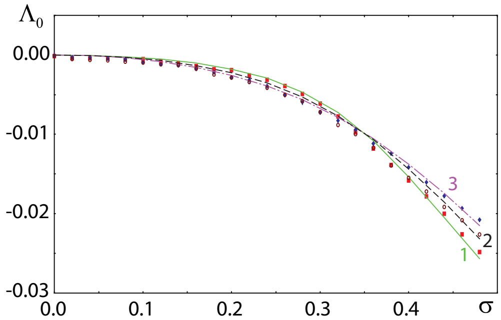

Fig. 4. The dependencies of ZLE Λ_0 on the control parameter σ for two Rössler oscillators (17). The data obtained numerically are represented by points, the lines correspond to the the theoretical predictions of ZLE (15). Curve 1 (solid line) corresponds to the noise intensity $B = 4 \times 10^2$, curve 2 (dashed line) represents Eq. (15) for $B = 6.5 \times 10^2$ and curve 3 (dotted line) $- B = 9 \times 10^2$.

shown by solid lines 1–3. Obviously, the results of numerical simulation of the behavior of two interacting Rössler systems with noise agree well with the theoretical data (16).

Since the distributions of the phase differences of the interacting chaotic systems with noise obtained numerically agree entirely with the theoretical expression, we can expect that the dependence of ZLE on the coupling parameter is also determined by the deduced Eq. (15). To check this assumption we compare three dependencies of ZLE on the coupling strength with the theoretical predictions calculated for the different values of the noise intensities.

Dependencies of ZLE Λ_0 on the coupling strength σ obtained by means of Benettin technique [2,3] for noise intensities $B = 4 \times 10^2$, $B = 6.5 \times 10^2$ and $B = 9 \times 10^2$ are given in Fig. 4 by points. Theoretical curves (15) are also given in Fig. 4 by lines, the parameters and variables are $f = 0.148$, $a_1 = 166.7$, $a_2 = 5.0$, $a_3 = 5.0$. Again, there is a good agreement between the data obtained numerically and the theoretical formula (15).

4. Conclusions

In conclusion, in the present paper we have reported for the first time on the analytical expression of ZLE for the coupled chaotic systems with noise. We have shown the full agreement between the analytical expression and the results obtained numerically for the coupled flow chaotic systems. Since the deduced theoretical formula is valid for the systems with random force as well as for the chaotic oscillators with noise, we believe that this approach can be applicable in different relevant circumstances, e.g. in physical or physiological systems. Indeed, many physiological and physical systems are known to demonstrate the phase synchronization regimes [57–64] or the pre-transitional behavior [32,35,65–68]. The found value of zero Lyapunov exponent may be used in these tasks, e.g., as the quantity characterizing the degree of the synchronization [69] to reveal the particularities of the system dynamics depending on the control parameter values.

Acknowledgments

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